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Gisin's theorem via Hardy's inequality

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ABSTRACT

The original Gisin's theorem states that all the entangled pure states of two qubits violate a single Bell's inequality, namely Clause-Horne-Shimony-Holt (CHSH) inequality. In this paper we show that all the entangled pure states for n particles also violate a single Bell's inequality, namely Hardy's inequality arising from Hardy's nonlocality test without inequality. Thus Gisin's theorem is proved in its most general form from which it follows that for pure states Bell's nonlocality and quantum entanglement are equivalent. In the sense of Gisin's theorem, Hardy's inequality can be regarded as a natural generalization of CHSH inequality.

1. Introduction

Quantum mechanics violates Bell's inequalities (Bell, 1964) which hold for all local hidden variable (LHV) theories. The violations of Bell's inequalities are intriguingly related to quantum entanglement. Gisin (1991) showed that all the entangled pure states of two qubits violate a single Bell's inequality, namely the Clause-Horne-Shimony-Holt (CHSH) inequality (Clauser *et al.*, 1969), with two different measurement settings for each observer. This equivalence between entanglement and nonlocality for pure states is referred to as Gisin's theorem. The first effort to generalize Gisin's theorem to multipartite systems was made by Popescu and Rohrlich (Popescu and Rohrlich, 1992; Cavalcanti *et al.*, 2011) who showed that all the entangled pure multipartite states violate a set of Bell inequalities. The central idea in their paper is that for any n-particle entangled states, there exists a local projection on a subset of $n-2$ particles that leaves the remaining two particles in an entangled state. Based on their idea, different states may require different inequalities, although they are the same type, namely CHSH type, but on different bipartite systems. A more interesting question is that whether or not we can find a *single* Bell inequality which is violated by all pure entangled states.

A natural candidate of Bell inequality for this purpose is the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality (Mermin, 1990; Ardehali, 1992; Belinskii and Klyshko, 1993), which is a kind of generalization of CHSH inequality to n qubits. However, for the following generalized n-qubit GHZ state given by

$$|\psi\rangle_{GHZ} = \cos\theta|0...0\rangle + \sin\theta|1...1\rangle, \tag{1}$$

Scarani and Gisin (2001) found that there is no violation to the MABK inequalities in the case of $\sin 2\theta \leq 1/\sqrt{2^{n-1}}$. We note that only *full correlations* are involved in MABK inequalities and a complete set of Bell's inequalities for full correlations is known as Werner-Wolf-Żukowski-Brukner (WWZB) inequalities (Żukowski and Brukner, 2002; Werner and Wolf, 2001). Here by complete it means the set of WWZB inequalities is a necessary and sufficient condition for a LHV description for full correlation functions in standard Bell-type experiments, in which each local observer can choose between two dichotomic observables. However even for this complete set Żukowski *et al.* (2002) showed that WWZB inequality cannot be violated by all entangled pure n-qubit states. Indeed in the case of odd number of qubits and $\sin 2\theta \leq 1/\sqrt{2^{n-1}}$, the above mentioned generalized GHZ state does not violate any n-particle correlation Bell inequality. A breakthrough was made by Chen *et al.* (2004) who showed numerically that all 3-qubit pure entangled states violate a Bell inequality for *probabilities*. As to higher dimensional systems, Gisin and Peres (1992) proved that all the entangled pure states of two qudits also violate the CHSH inequality and an alternative proof is given by Chen *et al.* (2008).

In (Hardy, 1992) he gave an argument of nonlocality without inequality for two particles and he showed (Hardy, 1993) that any pure entangled two-qubit state except maximally entangled states exhibits this type of nonlocality, namely, Hardy's nonlocality, which was extended to n particles by Pagonis and Clifton (1992). Based on Hardy's proof of nonlocality without inequality, Mermin formulated Hardy's inequality for two qubits (Mermin, 1994). Later Cereceda (2004) generalized Hardy's inequality to n qubits case, and found a rather interesting result that all entangled generalized GHZ state violates Hardy's inequality. By classifying the canonical form of three qubit state into different classes, Choudhary *et al.* (2010) gave an analytical proof for Gisin's theorem for three qubits by using Hardy's nonlocality and Hardy's inequality. Very recently, Wang and Markham (2012) showed the nonlocality for all entangled permutation symmetric states via Hardy's inequality. All these results indicate that Hardy's inequality is a promising candidate for proving Gisin's theorem.

Most recently, we (Yu *et al.*, 2012) have proved Gisin's theorem in its most general form by showing that all the entangled pure states violate a Hardy's inequality, which has two different measurement settings for each observer. In this paper, we give more details on the proof, especially we show the detailed proof for the general 2-qubit and 3-qubit entangled states as examples. Also we calculate the violation of Hardy's inequality on some important states such as generalized GHZ states, Dicke states. Moreover, we show how Hardy's nonlocality proof without inequality can be viewed as a state-dependent proof of quantum contextuality in the spirit of Kochen and Specker (1967).

This paper is organized as follows. In Section II, we give a brief introduction to Hardy's nonlocality proof and Hardy's inequality. Specially we shall construct a state-dependent proof of quantum locality out of Hardy's argument. In Section III, we prove that all entangled n-qubit states violate Hardy's inequality. By introducing the concepts of magic basis and magic subset for each given pure n-qubit state, we can divide all pure states into two overlapping subsets, namely Bell scenario and Hardy scenario. For each subset we make different ansatz on the measurement settings and manage to obtain the desired violation. In Section IV, we extend the proof for the case of n qudits. In Section V, we present some important examples such as generalized GHZ states, Dicke states, and general 2-qubit and 3-qubit entangled state etc. We end with a summary and some discussions.

2. Hardy's nonlocality and Hardy's inequality

Consider a system composed of n spacelike separated subsystems that are labeled with the index set $I = \{1, 2, \ldots, n\}$, and for each subsystem $k \in I$

we choose two observables $\{a_k, b_k\}$ taking binary values $\{0,1\}$. Hardy's argument for nonlocality without inequality claims that the following statements on probabilities of $n+2$ events

$$\langle \bar{b}_I\rangle_c = 0, \quad \langle b_k a_{\bar{k}}\rangle_c = 0 \quad (\forall k \in I), \quad \langle a_I\rangle_c > 0 \tag{2}$$

cannot hold simultaneously in any LHV theories. Here we have denoted $a_\alpha = \prod_{k\in\alpha} a_k$ and $\bar{b}_\alpha = \prod_{k\in\alpha} \bar{b}_k$ with $\bar{b}_k = 1 - b_k$ for an arbitrary subset $\alpha \subseteq I$ and $\bar{k} = I \setminus \{k\}$ for arbitrary $k \in I$. In any LHV theory the value, e.g., $a_k^\lambda = 0, 1$, of an observable, e.g., a_k, of a subsystem, is determined by some hidden variables λ and is independent of the choices of the observables of other subsystems. The hidden variables are distributed according to some probability ϱ_λ and, since we consider binary observables, the average of an observable, e.g., $\langle a_I\rangle_c := \int d\lambda \varrho_\lambda \prod_k a_k^\lambda$, is exactly the probability of finding all the observables a_k taking value 1.

Hardy's argument for nonlocality goes as follows. In any LHV theory from the first claim in Eq.(2) it follows that it is impossible to find all observables b_k with $k \in I$ taking value 0, meaning that there exists at least one $k \in I$ such that $b_k^\lambda = 1$ for some hidden variables λ with a nonzero measure. The second statement in Eq.(2) claims that for any given k and for any hidden variable λ (of measure 1) at least one observable among b_k and a_i with $i \neq k$ will take value 0. As a result for any hidden variable λ at least one observable among a_k with $k \in I$ will assume value 0 and the last statement in Eq.(2) cannot be true.

Hardy's nonlocality proof can also be regarded as a state-dependent proof of quantum contextuality for n qubits. Consider the following $(n+1)2^n + 1$ rays, i.e., one dimensional projectors

$$\begin{aligned}
\hat{r}_0 &= \bigotimes_{i\in I} |a_i\rangle\langle a_i|, \\
\hat{r}_{n+1} &= \bigotimes_{i\in I} |\bar{b}_i\rangle\langle \bar{b}_i|, \\
\hat{b}_T &= \bigotimes_{i\in I\setminus T} |b_i\rangle\langle b_i| \bigotimes_{i\in T} |\bar{b}_i\rangle\langle \bar{b}_i|, \quad T \subset I, \\
\hat{r}_{\alpha,k} &= \bigotimes_{i\in\alpha} |a_i\rangle\langle a_i| \bigotimes_{i\in\bar{k}\setminus\alpha} |\bar{a}_i\rangle\langle \bar{a}_i| \bigotimes |b_k\rangle\langle b_k|, \quad \alpha \subseteq \bar{k}, \\
\hat{t}_{\alpha,k} &= \bigotimes_{i\in\alpha} |a_i\rangle\langle a_i| \bigotimes_{i\in\bar{k}\setminus\alpha} |\bar{a}_i\rangle\langle \bar{a}_i| \bigotimes |\bar{b}_k\rangle\langle \bar{b}_k|, \quad \alpha \subseteq \bar{k},
\end{aligned} \tag{3}$$

together with an additional ray $\hat{r}$ that is orthogonal to all $\hat{r}_{\bar{k}k} (k \in I)$ and $\hat{r}_{n+1}$, and not orthogonal to $\hat{r}_0$. Such a ray can always be found as long as $\hat{r}_0$ is linearly independent of $\hat{r}_{\bar{k}k}(k \in I)$ and $\hat{r}_{n+1}$. Now we try to assign values $v : \hat{r} \mapsto v(\hat{r}) \in \{0, 1\}$ to the above $(n+1)2^n + 2$ rays according to the following KS value assignment rules:

1. Orthogonal rays are not assigned to value 1 simultaneously, i.e., $v(\hat{r})v(\hat{r}') = 0$ if $\hat{r} \perp \hat{r}'$;
2. In a basis there is always a ray that is assigned to value 1, i.e., $\sum_\alpha v(\hat{r}_\alpha) = 1$ if $\hat{r}_\alpha$ form a basis $\sum_\alpha \hat{r}_\alpha = I$ and $\hat{r}_\alpha \perp \hat{r}_\beta$ if $\alpha \neq \beta$.

In addition value 1 is preassigned to two rays $\hat{r}$ and $\hat{r}_0$, which is justified by the fact that they are nonorthogonal so that it is possible to preselect the system in the state $\hat{r}$ and postselect the system in the state $\hat{r}_0$. Then we can get a contradiction as follows. From $v(\hat{r}) = 1$ and $\hat{r} \perp r_{n+1}$, $\hat{r} \perp \hat{r}_{\bar{k}k}$ it follows that $v(\hat{r}_{n+1}) = 0$ and $v(\hat{r}_{\bar{k}k}) = 0$ for all $k \in I$ according to rule number 1. Similarly from $v(\hat{r}_0) = 1$ it follows that $v(\hat{r}_{\alpha,k}) = 0$ and $v(\hat{t}_{\alpha,k}) = 0$ for all $\alpha \subset \bar{k}, k \in I$. For a given $k \in I$, the set $\bigcup_{\alpha \subseteq \bar{k}}\{\hat{r}_{\alpha,k}, \hat{t}_{\alpha,k}\}$ forms a basis, taking account the above results $v(\hat{r}_{\bar{k},k}) = v(\hat{r}_{\alpha,k}) = v(\hat{t}_{\alpha,k}) = 0$ for any $\alpha \subset \bar{k}$ and rule number 2, we obtain $v(\hat{t}_{\bar{k},k}) = 1$ for any $k \in I$. According to KS rule 1 we deduce that $v(\hat{b}_T) = 0$ for all $T \subset I$ since $\hat{b}_T \perp \hat{t}_{\bar{k}k}$ for any $k \in T$. Since $\bigcup_{T \subset I} \hat{b}_T \bigcup \hat{r}_{n+1}$ forms a basis, finally we obtain $v(\hat{r}_{n+1}) = 1$, a contradiction. As a result in any noncontextual theory the Hardy's nonlocality relations (2) cannot hold simultaneously.

Interestingly, Hardy's nonlocality proof without inequality (2) also gives rise to a Bell inequality, which is referred to as Hardy's inequality. Bell's inequality is derived under the assumption of locality and realism. Equivalently it can be derived under the existence of a joint probability distribution of all the observables. Let us denote

$$H = a_I - \bar{b}_I - \sum_{k \in I} b_k a_{\bar{k}}. \tag{4}$$

and, since $a_I \bar{b}_I \leq \bar{b}_I$ and $a_k b_k \leq b_k$, we have

$$\langle H \rangle_c \leq \langle a_I(1 - \bar{b}_I - \sum_{k \in I} b_k) \rangle \leq 0 \tag{5}$$

with the last inequality ensured by the fact that $1 - \bar{b}_I = \vee_{k \in I} b_k \leq \sum_{k \in I} b_k$. Hardy's inequality is a Bell's inequality for *probabilities* and, as it stands, is applicable for a system of n particles each of which may have a different number of energy levels. For example, two particles and three particles Hardy's inequalities read,

$$\langle a_1a_2 - \bar{b}_1\bar{b}_2 - a_1b_2 - b_1a_2\rangle_c \leq 0 \tag{6}$$

$$\langle a_1a_2a_3 - \bar{b}_1\bar{b}_2\bar{b}_3 - a_1a_2b_3 - a_1b_2a_3 - b_1a_2a_3\rangle_c \leq 0 \tag{7}$$

Hardy's inequality can also be expressed in terms of correlation functions. If we let $A_i = 2a_i - 1, B_i = 2b_i - 1$ with $A_i, B_i \in \{-1, 1\}$, Eq. (6, 7) can be written as

$$\langle A_1A_2 - B_1B_2 - A_1B_2 - B_1A_2\rangle_c \leq 2, \tag{8}$$

$$\begin{aligned}&\langle A_1A_2A_3 + B_1B_2B_3 - A_1A_2B_3 - A_1B_2A_3 - B_1A_2A_3\rangle_c -\\ &\langle A_1B_2 + B_1A_2 + A_1B_3 + B_1A_3 + A_2B_3 + B_2A_3 + A_1 + A_2 + A_3\rangle_c\\ &\leq 3.\end{aligned} \tag{9}$$

For a given entangled pure state $|\psi\rangle$ of n particles, also labeled with I, to violate Hardy's inequality Eq. (4) one must find out two measurement settings $\{|a_k\rangle, |b_k\rangle\}$ for each particle $k \in I$ such that

$$\langle H\rangle_\psi := |\langle\psi|a_I\rangle|^2 - |\langle\psi|\bar{b}_I\rangle|^2 - \sum_{k\in I} |\langle\psi|a_{\bar{k}}b_k\rangle|^2 > 0, \tag{10}$$

where $|a_I\rangle = \otimes_{k\in I}|a_k\rangle_k$, $|\bar{b}_I\rangle = \otimes_{k\in I}|\bar{b}_k\rangle_k$ with $|\bar{b}_k\rangle_k$ being orthogonal to $|b_k\rangle_k$, and $|a_{\bar{k}}b_k\rangle = \otimes_{i\neq k}|a_i\rangle_i \otimes |b_k\rangle_k$. Hardy's nonlocality proof, in which the measurement settings are so chosen that only the first term of $\langle H\rangle_\psi$ is nonvanishing, provides a natural violation to Hardy's inequality. However not all entangled pure states, e.g., maximally entangled bipartite states (Hardy, 1993) and a subset of 3-qubit states (Wu and Xie, 1996), can have Hardy's nonlocality proof. On the other hand Hardy's inequality, being equivalent to the CHSH inequality in the case of two particles as seen from Eq. (8), is violated by all the entangled pure bipartite states (Gisin, 1991; Gisin and Peres, 1992). The analytical proof of Gisin's theorem for 3 qubits (Choudhary *et al.*, 2010) is also based on Hardy's inequality, which is found to be violated by all the entangled symmetric pure states of n qubits (Wang and Markham, 2012). Here we shall demonstrate that Hardy's inequality is violated by all entangled pure states.

3. Violations for qubits

In this section we prove that Hardy's inequality is violated by all entangled pure states for qubits. Firstly we introduce two useful concepts, magic basis and magic subset for a given pure entangled state. According to the magic subsets all the pure states fall into one or two subsets called Bell scenario and Hardy scenario. For each scenario we manage to find out the measurement settings that will lead to the violation of Hardy's inequality.

3.1 Magic basis and magic subset

We define a computational basis $\{|0_\alpha 1_{\bar\alpha}\rangle\}_{\alpha\subseteq I}$ of n qubits to be a *magic basis* for $|\psi\rangle$, if the coefficients $h_\alpha = \langle\psi|0_\alpha 1_{\bar\alpha}\rangle$ in the expansion

$$|\psi\rangle = \sum_{\alpha\subseteq I} h_\alpha^* |0_\alpha 1_{\bar\alpha}\rangle \tag{11}$$

satisfy the conditions

1. $h_I \neq 0$
2. $h_{\bar k} = 0$ for all $k \in I$ with $\bar k = I \setminus \{k\}$.

For example, $\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |011\rangle, |101\rangle, |110\rangle, |111\rangle\}$ forms a magic basis for generalized GHZ state $\cos\theta|000\rangle + \sin\theta|111\rangle$ while does not form a magic basis for the state $(|000\rangle + |001\rangle + |010\rangle)/\sqrt{3}$. As to n qubits example, $\{|0_\alpha 1_{\bar\alpha}\rangle\}_{\alpha\subseteq I}$ forms a magic basis for generalized GHZ state $\cos\theta|0_I\rangle + \sin\theta|1_I\rangle$ but not a magic basis for the state $\propto |0_I\rangle + \sum_{k\in I} |0_{\bar k} 1_k\rangle$.

By a suitable choice of the local basis for each qubit, a magic basis can always be found. For example we can construct a magic basis for a given pure state $|\psi\rangle$ with the help of its *closest product state* $|p_I\rangle = \otimes_{k\in I}|p_k\rangle_k$ whose inner product with $|\psi\rangle$ is the largest among all possible product states. The closest product state always exists, albeit difficult to find, and makes the definition of the geometric measure of entanglement (Wei and Goldbart, 2003) possible. Let $|\bar p_k\rangle_k$ be the state orthogonal to $|p_k\rangle_k$ for each qubit $k \in I$, then $\{|p_\alpha \bar p_{\bar\alpha}\rangle\}_{\alpha\subseteq I}$ is a magic basis for $|\psi\rangle$. This is because $|h_I|^2 > 0$ and if there were a $k \in I$ such that $h_{\bar k} = \langle\psi|p_{\bar k}\bar p_k\rangle \neq 0$ then, by introducing a normalized single qubit state $|\phi\rangle_k \propto h_I^*|p_k\rangle_k + h_{\bar k}^*|\bar p_k\rangle_k$, we would have $|\langle\psi|p_{\bar k}\phi_k\rangle|^2 = |h_{\bar k}|^2 + |h_I|^2 > |h_I|^2$, which contradicts the definition of the closest product state as $|p_{\bar k}\phi_k\rangle$ is a product state. The magic basis for a bipartite state coincides with its Schmidt decomposition. In general the magic basis for a given pure state is not unique and the one obtained from the closest product state only provides us a possibility.

Under a magic basis for an entangled pure state $|\psi\rangle$ there is at least one $\alpha \subset I$ such that $h_\alpha \neq 0$ and we introduce a nonnegative integer

$$m = \max_{\alpha\in\mathcal{C}} |\alpha|, \quad \mathcal{C} = \{\alpha \subset I | h_\alpha \neq 0\} \tag{12}$$

for each pure state. We refer to a subset $A \in \mathcal{C}$ with $|A| = m$ as a *magic subset* for $|\psi\rangle$, which may not be unique. For a magic subset A it holds $h_A \neq 0$ while

	a_{k0}	a_{k1}	b_{k0}	b_{k1}	$\bar{b}_{k0}$	$\bar{b}_{k1}$	
$k \in A$	1	0	$-\sin\gamma$	$\cos\gamma$	$\cos\gamma$	$\sin\gamma$	$m = n-2$
$k \in \bar{A}$	1	0	q	r	$-r^*$	q	
$k \in A$	1	0	c_k	$z-1$	$1-z^*$	c_k^*	$m < n-2$
$k \in S$	1	y	1	yz	$-yz^*$	1	
$k = v$	e	1	-1	f	f^*	1	

Table 1: Two measurement settings $|a_k\rangle = a_{k0}|0\rangle_k + a_{k1}|1\rangle_k$ and $|b_k\rangle = b_{k0}|0\rangle_k + b_{k1}|1\rangle_k$ for each particle $k \in I$ for pure states in the Bell scenario $m = n-2$ (upper half) and the Hardy scenario (lower half) $m < n-2$. For each $k \in I$ state $|\bar{b}_k\rangle_k = \bar{b}_{k0}|0\rangle_k + \bar{b}_{k1}|1\rangle_k$ is orthogonal to $|b_k\rangle_k$.

$h_B = 0$ for any $B \subset I$ with $|A| < |B| < n$. On the other hand, in a magic basis of $|\psi\rangle$ if the collection $\mathcal{C}$ is not empty then the state is entangled because local projection to $|0\rangle_k$ to each qubit k in a magic subset A will leave those qubits in $\bar{A}$ in a generalized GHZ state with nonzero coefficients h_I and h_A, which is obviously entangled. By the definition of the magic basis we have $m \le n-2$. For example for the generalized GHZ state $\cos\theta|000\rangle + \sin\theta|111\rangle (0 < \theta < \pi/2)$, one obtains $\mathcal{C} = \{\emptyset\}, m = 0$, thus A is the empty set $\emptyset$. Another example is that for the state $(|000\rangle + |011\rangle + |101\rangle + |111\rangle)/2$, one gets $\mathcal{C} = \{\emptyset, \{1\}, \{2\}\}, m = 1$, thus A can be $\{1\}$ or $\{2\}$.

3.2 Bell scenario

If $m = n-2$, i.e., there exists $A \subset I$ such that $h_A \ne 0$ with $|A| = n-2$ and $I = A \cup \bar{A}$, we refer to this case as the Bell scenario. In the upper part of Table.I two measurement settings $\{|a_k\rangle, |b_k\rangle\}$ are specified for each qubit $k \in I$ with qubits in A or $\bar{A}$ having the same pair of measurement settings, where

$$|a_I\rangle = \bigotimes_{k\in I} |a_k\rangle_k = |0_I\rangle \tag{13}$$

$$|a_{\bar{k}} b_k\rangle = \bigotimes_{l\in\bar{k}} |a_l\rangle_l \otimes |b_k\rangle_k = \begin{cases} -\sin\gamma|0_I\rangle + \cos\gamma|0_{\bar{k}}1_k\rangle & (k \in A) \\ q|0_I\rangle + r|0_{\bar{k}}1_k\rangle & (k \in \bar{A}) \end{cases} \tag{14}$$

$$\begin{aligned} |\bar{b}_I\rangle &= \bigotimes_{k\in I} |\bar{b}_k\rangle_k = \bigotimes_{k\in A}\Big(\cos\gamma|0\rangle_k + \sin\gamma|1\rangle_k\Big)\bigotimes_{l\in\bar{A}}\Big(-r^*|0\rangle_l + q|1\rangle_l\Big) \\ &= \sum_{\beta\subseteq A,\alpha\subseteq\bar{A}} (\cos\gamma)^{|\beta|}(\sin\gamma)^{m-|\beta|}(-r^*)^{|\alpha|}q^{2-|\alpha|}|0_{\beta\cup\alpha}1_{\overline{\beta\cup\alpha}}\rangle. \end{aligned} \tag{15}$$

As a result we have $\langle\psi|a_I\rangle = h_I$ and $\langle\psi|a_{\bar{k}}b_k\rangle = -\sin\gamma h_I$ if $k \in A$ while $\langle\psi|a_{\bar{k}}b_k\rangle = qh_I$ if $k \in \bar{A}$, where we have used the fact that in the magic basis $h_{\bar{k}} = \langle\psi|0_{\bar{k}}1_k\rangle = 0$ for arbitrary $k \in I$. For the sake of convenience, we define $q = \sqrt{\lambda}/(1+\lambda)$ and $r = ie^{-i\theta/2}\sqrt{1-q^2}$ with $\lambda = |h_A/h_I| > 0$ and $h_A/h_I = \lambda e^{i\theta}$. Taking into account these definitions in addition to the fact that $h_{\bar{k}} = 0$ for all $k \in I$ and $\bar{A}$ has only two elements, we obtain

$$\langle H\rangle_\psi = |h_I|^2(1 - 2q^2 - (n-2)\sin^2\gamma) - |\langle\psi|\bar{b}_I\rangle|^2, \tag{16}$$

where

$$\begin{aligned}
\langle\psi|\bar{b}_I\rangle &= \Big(h_I(-r^*)^2 + q^2 h_A\Big)\cos^m\gamma \\
&\quad + \sum_{\beta\subset A,\alpha\subseteq\bar{A}} (\cos\gamma)^{|\beta|}(\sin\gamma)^{m-|\beta|}(-r^*)^{|\alpha|}q^{2-|\alpha|}h_{\beta\cup\alpha} \\
&= -\frac{h_I e^{i\theta}}{1+\lambda}\cos^m\gamma + \sum_{k=1}^{m}(\cos\gamma)^{m-k}(\sin\gamma)^k \\
&\quad \sum_{\beta\subset A,|\beta|=m-k}\left((-r^*)^2 h_{\beta\cup\bar{A}} - r^* q\sum_{v\in\bar{A}} h_{\beta\cup v} + q^2 h_\beta\right).
\end{aligned} \tag{17}$$

If $\gamma = 0$ we already have a violation

$$\langle H\rangle_\psi|_{\gamma=0} = |h_A h_I|^2/(|h_A| + |h_I|)^2 > 0, \tag{18}$$

and in this case two measurement directions for qubits in A become identical. To have a nondegenerate pair of measurement settings we notice that $\langle H\rangle_\psi$ is a continuous function of γ and there exists some small $\epsilon \neq 0$ such that $\langle H\rangle_\psi|_{\gamma=\epsilon} > 0$.

For example, consider three-qubit states, in Bell scenario the states can be written as $|\psi\rangle = h_I^*|000\rangle + h_A^*|011\rangle + \cdots$, and $|a_I\rangle$, $|a_{\bar{k}}b_k\rangle$ and $|\bar{b}_I\rangle$ are as follows:

$$\begin{aligned}
|aaa\rangle &= |000\rangle, \\
|aab\rangle &= |00\rangle \otimes (q|0\rangle + r|1\rangle), \\
|aba\rangle &= |0\rangle \otimes (q|0\rangle + r|1\rangle) \otimes |0\rangle, \\
|baa\rangle &= (-\sin\gamma|0\rangle + \cos\gamma|1\rangle) \otimes |00\rangle, \\
|\bar{b}\bar{b}\bar{b}\rangle &= (\cos\gamma|0\rangle + \sin\gamma|1\rangle) \otimes (-r^*|0\rangle + q|1\rangle) \otimes (-r^*|0\rangle + q|1\rangle).
\end{aligned}$$

If $\gamma = 0$ we have:

$$\begin{aligned}
\langle\psi|aaa\rangle &= h_I, \\
\langle\psi|aab\rangle &= h_I q, \\
\langle\psi|aba\rangle &= h_I q, \\
\langle\psi|baa\rangle &= 0, \\
\langle\psi|\bar{b}\bar{b}\bar{b}\rangle &= -e^{i\theta}\frac{h_I}{1+\lambda}
\end{aligned} \tag{19}$$

Therefore,

$$\langle H\rangle_\psi|_{\gamma=0} = |h_I|^2(1-2q^2) - \frac{|h_I|^2}{(1+\lambda)^2} = \frac{|h_A h_I|^2}{(|h_A|+|h_I|)^2} > 0. \tag{20}$$

3.3 Hardy scenario

If $m < n-2$, i.e., there exists $A \subset I$ with $|A| = m$ such that $h_A \neq 0$ while $h_B = 0$ if $m < |B| < n$, we refer to this case as the Hardy scenario because the state exhibits Hardy-type nonlocality: the measurement settings can be so chosen that only the first term in Eq. (10) is nonzero. Consider the partition of the index set I into 3 disjoint subsets $I = A \cup S \cup \{v\}$ with an arbitrary $v \in \bar{A}$ and $|S| = s = n - m - 1 \geq 2$.

First of all, we shall show that $\langle\psi|a_I\rangle = y^s h_A(1-z)$. According to the lower part of Table. I we have

$$\begin{aligned}
|a_I\rangle &= \bigotimes_{k\in I}|a_k\rangle_k \\
&= |0_A\rangle\bigotimes_{k\in S}\Big(|0\rangle_k + y|1\rangle_k\Big)\otimes\Big(e|0\rangle_v + |1\rangle_v\Big) \\
&= \sum_{\beta\subseteq S} y^{|\beta|}|0_{\overline{\beta\cup v}}1_{\beta\cup v}\rangle + e\sum_{\beta\subseteq S} y^{|\beta|}|0_{\bar\beta}1_\beta\rangle
\end{aligned} \tag{21}$$

with normalizations neglected, determined by a real parameter $y \neq 0$ and a complex parameter $z \neq 1$ together with $e = -h_A y^s z/h_I$. Because $A \subset \overline{\beta\cup v} \neq I$ for $\beta \subset S$ we have $n > |\overline{\beta\cup v}| > |A| = m$ and thus $h_{\overline{\beta\cup v}} = 0$ if $\beta \neq S$ while $h_{\overline{S\cup v}} = h_A$. Because $A \subset \bar\beta \neq I$ for any nonempty $\beta \subseteq S$ we have $n > |\bar\beta| > |A| = m$ if β is not empty and thus $h_{\bar\beta} = 0$ if $\beta \neq \emptyset$ while $h_{\bar\emptyset} = h_I$. Considering $e = -zy^s h_A/h_I$ we obtain

$$\langle\psi|a_I\rangle = y^s h_A + e h_I = y^s h_A(1-z). \tag{22}$$

Secondly, we shall show that $\langle\psi|a_{\bar{k}}b_k\rangle = 0$ for all $k \in I = A \cup S \cup v$.

I) If $k = v$ then

$$\begin{aligned} |a_{\bar{v}}b_v\rangle &= \bigotimes_{l\in\bar{v}} |a_l\rangle_l \otimes |b_v\rangle_v \\ &= |0_A\rangle \bigotimes_{l\in S} \Big(|0\rangle_l + y|1\rangle_l\Big) \otimes \Big(-|0\rangle_v + f|1\rangle_v\Big) \\ &= \sum_{\beta\subseteq S} y^{|\beta|}\Big(f|0_{\overline{\beta\cup v}}1_{\beta\cup v}\rangle - |0_{\bar{\beta}}1_\beta\rangle\Big). \end{aligned} \tag{23}$$

Since $h_{\overline{\beta\cup v}} = 0$ for arbitrary $\beta \subset S$ and $h_{\bar{\beta}} = 0$ for arbitrary nonempty $\beta \subseteq S$ we obtain

$$\langle\psi|a_{\bar{v}}b_v\rangle = y^s h_A f - h_I = 0, \tag{24}$$

i.e.,

$$f = h_I y^{-s}/h_A. \tag{25}$$

II) If $k \in A$ we have

$$\begin{aligned} |a_{\bar{k}}b_k\rangle &= \bigotimes_{l\in\bar{k}} |a_l\rangle_l \otimes |b_k\rangle_k = |0_{A\setminus k}\rangle \otimes \Big(c_k|0\rangle_k + (z-1)|1\rangle_k\Big) \\ &\quad \bigotimes_{k\in S} \Big(|0\rangle_k + y|1\rangle_k\Big) \otimes \Big(e|0\rangle_v + |1\rangle_v\Big) \\ &= c_k|a_I\rangle + \\ &\quad (z-1)\sum_{\beta\subseteq S} y^{|\beta|}\Big(|0_{\overline{\beta\cup\{k,v\}}}1_{\beta\cup\{k,v\}}\rangle + e|0_{\overline{\beta\cup k}}1_{\beta\cup k}\rangle\Big) \end{aligned} \tag{26}$$

If $|\beta| < s$ we have $|\overline{\beta\cup k}| = n - |\beta| - 1 > n - s - 1 = m$ and thus $h_{\overline{\beta\cup k}} = 0$. If $|\beta| < s-1$ we have $|\overline{\beta\cup\{k,v\}}| = n - |\beta| - 2 > m$ and thus $h_{\overline{\beta\cup\{k,v\}}} = 0$. As a result

$$\begin{aligned} \langle\psi|a_{\bar{k}}b_k\rangle &= c_k\langle\psi|a_I\rangle + (z-1)\sum_{\beta\subseteq S} y^{|\beta|}\Big(h_{\overline{\beta\cup\{k,v\}}} + eh_{\overline{\beta\cup k}}\Big) \\ &= c_k y^s h_A(1-z) + (z-1)y^s\Big(h_{\overline{S\cup\{k,v\}}} + eh_{\overline{S\cup k}}\Big) \\ &\quad +(z-1)y^{s-1}\sum_{\beta\subset S,|\beta|=s-1} h_{\overline{\beta\cup\{k,v\}}} \\ &= y^s h_A(1-z)\left(c_k - \frac{h_{A\setminus k}}{h_A} - \frac{eh_{(A\setminus k)\cup v}}{h_A} - \sum_{k'\in S}\frac{h_{(A\setminus k)\cup k'}}{yh_A}\right) \\ &= 0, \end{aligned} \tag{27}$$

where we have used the fact that $\beta \subset S$ with $|\beta| = s-1$ is equivalent to $\beta = S \setminus k'$ with $k' \in S$. Therefore,

$$c_k = \sum_{k' \in S} \frac{h_{(A\setminus k)\cup k'}}{y h_A} + \frac{h_{A\setminus k}}{h_A} - y^s \frac{h_{(A\setminus k)\cup v}}{h_I} z \tag{28}$$

for $k \in A$.

III) If $k \in S$ we have (from the lower part of Table I)

$$\begin{aligned} |a_{\bar{k}} b_k\rangle &= \bigotimes_{l\in\bar{k}} |a_l\rangle_l \otimes |b_k\rangle_k = |0_A\rangle \otimes \Big(|0\rangle_k + yz|1\rangle_k\Big) \\ &\quad \bigotimes_{l\in S\setminus k} \Big(|0\rangle_l + y|1\rangle_l\Big) \otimes \Big(e|0\rangle_v + |1\rangle_v\Big) \\ &= |a_I\rangle + \\ &\quad y(z-1) \sum_{\beta\subseteq S\setminus k} y^{|\beta|} \Big(e|0_{\overline{\beta\cup k}} 1_{\beta\cup k}\rangle + |0_{\overline{\beta\cup\{k,v\}}} 1_{\beta\cup\{k,v\}}\rangle\Big). \end{aligned} \tag{29}$$

Since $\beta \subseteq S \setminus k$ we have $|\beta| \le s-1$ and thus $h_{\overline{\beta\cup k}} = 0$ for all $\beta \subseteq S\setminus k$ and $h_{\overline{\beta\cup\{k,v\}}} = 0$ for all $\beta \subset S \setminus k$. As a result

$$\langle\psi|a_{\bar{k}} b_k\rangle = \langle\psi|a_I\rangle + y^s (z-1) h_A = 0. \tag{30}$$

Thirdly, according to the lower part of Table I we have

$$\begin{aligned} \langle \bar{b}_I| &= \bigotimes_{k\in I} \langle \bar{b}_k|_k = \bigotimes_{k\in A} \Big(\langle 0|_k (1-z) + \langle 1|_k c_k\Big) \\ &\quad \bigotimes_{l\in S} \Big(-\langle 0|_l yz + \langle 1|_l\Big) \otimes \Big(\langle 0|_v f + \langle 1|_v\Big) \\ &= \sum_{\alpha\subseteq A, \beta\subseteq S} \Big(\langle 0_{\alpha\cup\beta} 1_{\overline{\alpha\cup\beta}}| + \langle 0_{\alpha\cup\beta\cup v} 1_{\overline{\alpha\cup\beta\cup v}}| f\Big) \\ &\quad (-yz)^{|\beta|} (1-z)^{|\alpha|} \prod_{k\in A\setminus\alpha} c_k \end{aligned} \tag{31}$$

and therefore

$$\begin{aligned} \langle \bar{b}_I|\psi\rangle &= \sum_{\alpha\subseteq A, \beta\subseteq S} \Big(h^*_{\alpha\cup\beta} + f h^*_{\alpha\cup\beta\cup v}\Big)(-yz)^{|\beta|}(1-z)^{|\alpha|} \prod_{k\in A\setminus\alpha} c_k \\ &= \sum_{\beta\subseteq S} \Big(h^*_{A\cup\beta} + f h^*_{A\cup\beta\cup v}\Big)(-yz)^{|\beta|}(1-z)^m \\ &\quad + \sum_{\alpha\subset A, \beta\subseteq S} \Big(h^*_{\alpha\cup\beta} + f h^*_{\alpha\cup\beta\cup v}\Big)(-yz)^{|\beta|}(1-z)^{|\alpha|} \prod_{k\in A\setminus\alpha} c_k \end{aligned}$$

$$\begin{aligned}
&= \left(h_A^* + (-yz)^s f h_I^*\right)(1-z)^m \\
&\quad + \sum_{\alpha\subset A,\beta\subseteq S} \left(h^*_{\alpha\cup\beta} + \frac{h_I h^*_{\alpha\cup\beta\cup v}}{y^s h_A}\right)(-yz)^{|\beta|}(1-z)^{|\alpha|} \prod_{k\in A\setminus\alpha} c_k \\
&= \left(h_A^* + (-z)^s \frac{|h_I|^2}{h_A}\right)(1-z)^m \\
&\quad + \sum_{\alpha\subset A,\beta\subseteq S} \sum_{\sigma=0}^{1} G^{(\sigma)}_{\alpha\beta} y^{-\sigma s}(-yz)^{|\beta|}(1-z)^{|\alpha|} \prod_{k\in A\setminus\alpha} c_k \qquad (32)
\end{aligned}$$

where in the third equality we have used the fact that for $\beta \subseteq S$ $h_{A\cup\beta} \neq 0$ if and only if β is empty while $h_{A\cup\beta\cup v} \neq 0$ if and only if $\beta = S$ and in the fourth equality we have introduced $G^{(0)}_{\alpha\beta} = h^*_{\alpha\cup\beta}$ and $G^{(1)}_{\alpha\beta} = h_I h^*_{\alpha\cup\beta\cup v}/h_A$. To proceed we calculate

$$\begin{aligned}
\prod_{k\in A\setminus\alpha} c_k &= \prod_{k\in A\setminus\alpha}\left(\sum_{k'\in S}\frac{h_{(A\setminus k)\cup k'}}{y h_A} + \frac{h_{A\setminus k}}{h_A} - y^s \frac{h_{(A\setminus k)\cup v}}{h_I} z\right) \\
&= \sum_{\substack{\omega_1,\omega_2\subseteq A\setminus\alpha \\ \omega_1\cap\omega_2=\emptyset}} \left(\prod_{k\in(A\setminus\alpha)\setminus(\omega_1\cup\omega_2)} \sum_{k'\in S}\frac{h_{(A\setminus k)\cup k'}}{h_A}\right) \\
&\quad \left(\prod_{k\in\omega_1}\frac{h_{(A\setminus k)\cup v}}{h_I}\right)\left(\prod_{k\in\omega_2}\frac{h_{A\setminus k}}{h_A}\right)\frac{(-y^s z)^{|\omega_1|}}{y^{|A\setminus\alpha|-|\omega_1\cup\omega_2|}}. \qquad (33)
\end{aligned}$$

Suppose $|\omega_1| = t$, $|\omega_2| = i$, $|\beta| = p$, and $|\alpha| = m-u$. Because $\beta \subseteq S$ we have $0 \le p \le s$ and because $h_{\alpha\cup\beta} = 0$ if $|\alpha\cup\beta| > m$ we have $p + m - u \le m$, i.e., $p \le u$. Furthermore we have $0 \le i \le u$ and $0 \le t \le u - i$ due to the facts that $\omega_2 \subseteq A \setminus \alpha$, $\omega_1 \cap \omega_2 = \emptyset$, and $\omega_2 \subseteq A \setminus \alpha$). Moreover since $\alpha \subset A$ we have $u \neq 0$. We denote $D = \{(\sigma, p, u, i, t) | 0 \le p \le s, p \le u \le m, 0 \le i \le u, 0 \le t \le u - i, \sigma = 0, 1\}$. By substituting Eq.(33) into Eq.(32) we obtain

$$\begin{aligned}
&\langle \bar{b}_I|\psi\rangle - \left(h_A^* + (-z)^s\frac{|h_I|^2}{h_A}\right)(1-z)^m \\
= &\sum_{(\sigma,p,u,i,t)\in D\cap\{u\neq 0\}} y^{t(s+1)+p+i-u-\sigma s}(-z)^{t+p}(1-z)^{m-u} \\
&\sum_{\substack{\alpha\subseteq A \\ |\alpha|=m-u}} \sum_{\substack{\beta\subseteq S \\ |\beta|=p}} G^{(\sigma)}_{\alpha\beta} \sum_{\substack{\omega_1,\omega_2\subseteq A-\alpha,\omega_1\cap\omega_2=\emptyset \\ |\omega_1|=t,|\omega_2|=i}} \left(\prod_{k\in(A\setminus\alpha)\setminus(\omega_1\cup\omega_2)}\sum_{k'\in S}\frac{h_{(A\setminus k)\cup k'}}{h_A}\right) \\
&\left(\prod_{k\in\omega_1}\frac{h_{(A\setminus k)\cup v}}{h_I}\right)\left(\prod_{k\in\omega_2}\frac{h_{A\setminus k}}{h_A}\right)
\end{aligned}$$

$$:= \sum_{(\sigma,p,u,i,t)\in D\cap\{u\neq 0\}} y^{t(s+1)+p+i-u-\sigma s}(-z)^{t+p}(1-z)^{m-u}\ \Gamma^{(\sigma)}_{puit}. \tag{34}$$

From the fact that $u = 0$ leads to $p = t = i = 0$ and $\Gamma^{(0)}_{0000} = h^*_A$, $\Gamma^{(1)}_{0000} = 0$, we have

$$\begin{aligned}
\langle \bar b_I|\psi\rangle &= \Big(h^*_A + h^*_I(-yz)^s f\Big)(1-z)^m \\
&\quad + \sum_{\alpha\subset A,\beta\subseteq S}(-yz)^{|\beta|}(1-z)^{|\alpha|}\left(h^*_{\alpha\cup\beta} + f h^*_{\alpha\cup\beta\cup v}\right)\prod_{k\in A-\alpha} c_k \\
&= \frac{|h_I|^2}{h_A}(-z)^s(1-z)^m \\
&\quad + \sum_{(\sigma,p,u,i,t)\in D} y^{t(s+1)+p+i-u-\sigma s}\ \Gamma^{(\sigma)}_{puit}(-z)^{t+p}(1-z)^{m-u} \\
&:= \sum_{k=-m-s}^{(m+1)s} y^k L_k(z),
\end{aligned} \tag{35}$$

where we have denoted

$$\begin{aligned}
\Gamma^{(\sigma)}_{puit} &= \sum_{\substack{\alpha\subseteq A\\ |\alpha|=m-u}}\ \sum_{\substack{\beta\subseteq S\\ |\beta|=p}} G^{(\sigma)}_{\alpha\beta} \sum_{\substack{\omega_1,\omega_2\subseteq A\setminus\alpha,\omega_1\cap\omega_2=\emptyset\\ |\omega_1|=t,|\omega_2|=i}} \left(\prod_{k\in A\setminus(\alpha\cup\omega_1\cup\omega_2)}\ \sum_{k'\in S}\frac{h_{(A\setminus k)\cup k'}}{h_A}\right) \\
&\quad \left(\prod_{k\in\omega_1}\frac{h_{(A\setminus k)\cup v}}{h_I}\right)\left(\prod_{k\in\omega_2}\frac{h_{A\setminus k}}{h_A}\right)
\end{aligned} \tag{36}$$

with $G^{(0)}_{\alpha\beta} = h^*_{\alpha\cup\beta}$ and $G^{(1)}_{\alpha\beta} = h_I h^*_{\alpha\cup\beta\cup v}/h_A$. If we denote $D_0 = \{(\sigma,p,u,i,t)\in D|u - i + \sigma s = t(s+1) + p\}$ then

$$\begin{aligned}
L_0(z) &= \frac{|h_I|^2}{h_A}(-z)^s(1-z)^m \\
&\quad + \sum_{(\sigma,p,u,i,t)\in D_0}\Gamma^{(\sigma)}_{puit}(-z)^{t+p}(1-z)^{m-u} := \sum_{k=0}^{n-1} l_k(-z)^k.
\end{aligned} \tag{37}$$

Finally we note that, since $t+i \le u \le m$, $p \le s$, and $t, p, i \ge 0$, we have bounds

$$-m-s \le -u-s \le t(s+1)+p+i-u-\sigma s \le ts+p \le (m+1)s \tag{38}$$

with upper and lower bounds attained by $\{t = u = m, i = \sigma = 0, p = s\}$ and $\{t = p = i = 0, u = m, \sigma = 1\}$, respectively.

We shall prove via *reductio ad absurdum* that there exists nonzero $y = y_0$ such that the algebraic equation $\langle \bar b_I|\psi\rangle = 0$ of z has one root $z = z_0 \neq 1$. If all the roots of $\langle \bar{b}_I|\psi\rangle = 0$ were equal to 1 for any $y \neq 0$, then $\langle \bar{b}_I|\psi\rangle$ as a polynomial of z of degree $m + s = n - 1$ would be proportional to $(1 - z)^{n-1}$ and thus all the coefficients $L_k(z)$, especially $L_0(z)$, would be proportional to $(1-z)^{n-1}$ since $\{y^n\}_{n=-\infty}^{\infty}$ are linearly independent. On the one hand we have $l_{n-1} = |h_I|^2/h_A$ and $l_{n-2} = ml_{n-1}$ for $L_0(z)$, taking into account the facts that $n-2 > m$ and the sum term in Eq. (37) as a polynomial of z is of degree at most m because $t+p \leq u$ in D_0 since $u \geq p$ for $t = 0$ and $u-t-p = i+(t-\sigma)s \geq 0$ with $\sigma = 0, 1$ for $t \geq 1$. On the other hand for $(1 - z)^{n-1} := \sum_{k=0}^{n-1} l'_k(-z)^k$ we have $l'_{n-2}/l'_{n-1} = n - 1 > m = l_{n-2}/l_{n-1}$, a contradiction.

Taking into account the normalization of $|a_I\rangle$ and parameters y_0 and z_0 determined above we can obtain the desired violation

$$\langle H\rangle_\psi = \frac{|y_0^s h_A h_I(1 - z_0)|^2}{(1 + y_0^2)^s\,(|h_I|^2 + |y_0^s h_A z_0|^2)} > 0. \tag{39}$$

Two measurements directions in Table.I may become identical for a qubit $k \in A$ if $c_k = 0$, for all the qubits in S if $z^*y^2 = -1$, and for qubit v if $f^* = e$, i.e., $-|h_I|^2 = |h_A|^2y^2sz$. In these cases the degeneracy can be avoided by replacing b_{k0} with $b_{k0}+x$, where x is a real variable, while keeping y_0 and z_0 unchanged. Since $\langle H\rangle_\psi$ depends on x continuously and $\langle H\rangle_\psi|_{x=0} > 0$, there exists small ϵ such that $\langle H\rangle_\psi|_{x=\epsilon} > 0$ while two measurement directions are different for every qubit.

For example, consider three-qubit states, in Hardy's scenario the states can be written as $|\psi\rangle = h_I^*|000\rangle + h_\emptyset^*|111\rangle$ with $h_\emptyset \neq 0$. Let $y_0 = 1$ we have the following unnormalized measurement settings:

$$\begin{aligned}
|aaa\rangle &= (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (-\frac{h_\emptyset z}{h_I}|0\rangle + |1\rangle), \\
|aab\rangle &= (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (-|0\rangle + \frac{h_I}{h_\emptyset}|1\rangle), \\
|aba\rangle &= (|0\rangle + |1\rangle) \otimes (|0\rangle + z|1\rangle) \otimes (-\frac{h_\emptyset z}{h_I}|0\rangle + |1\rangle), \\
|baa\rangle &= (|0\rangle + z|1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (-\frac{h_\emptyset z}{h_I}|0\rangle + |1\rangle), \\
|\bar{b}\bar{b}\bar{b}\rangle &= (-z^*|0\rangle + |1\rangle) \otimes (-z^*|0\rangle + |1\rangle) \otimes (\frac{h_I^*}{h_\emptyset^*}|0\rangle + |1\rangle).
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\langle\psi|aaa\rangle &= h_I(-\frac{h_\emptyset}{h_I})z + h_\emptyset = h_\emptyset(1-z),\\
\langle\psi|aab\rangle &= -h_I + h_\emptyset\frac{h_I}{h_\emptyset} = 0,\\
\langle\psi|aba\rangle &= -h_I\frac{h_\emptyset}{h_I}z + h_\emptyset z = 0,\\
\langle\psi|baa\rangle &= -h_I\frac{h_\emptyset}{h_I}z + h_\emptyset z = 0,\\
\langle\psi|\bar{b}\bar{b}\bar{b}\rangle &= h_I(z^*)^2\frac{h_I^*}{h_\emptyset^*} + h_\emptyset. \qquad (40)
\end{aligned}$$

In order for $\langle\psi|aaa\rangle \neq 0$ and $\langle\psi|\bar{b}\bar{b}\bar{b}\rangle = 0$, we can choose $z_0 = -i|h_\emptyset|/|h_I|$. Thus, we have a violation of Hardy's inequality.

To sum up, for a given entangled pure n-qubit state $|\psi\rangle$ to violate Hardy's inequality we need only to find a magic basis and a magic subset A for $|\psi\rangle$ and choose one set of the measurements defined in Table 1 according to whether $|A| = m$ equals to $n-2$ or not.

4. Violation for qudits

Now we consider n qudits, also labeled with I, each of which may have a different number of energy levels. For a given pure n-qudit state $|\psi\rangle$ a magic basis can be defined similarly as in the case of qubits from its closest product state $|p_I\rangle = \otimes_{k\in I}|p_k\rangle_k$ satisfying $|\langle\psi|p\rangle|^2 \leq |\langle\psi|p_I\rangle|^2$ for any product state $|p\rangle$. We denote by $\mathcal{C}$ the collection of $\alpha \subset I$ such that for each $k \in \bar{\alpha}$ there exists a qudit state $|\bar{p}_k\rangle_k$ orthogonal to $|p_k\rangle_k$ such that $\langle\psi|p_\alpha\bar{p}_{\bar{\alpha}}\rangle \neq 0$. As long as $|\psi\rangle$ is entangled the collection $\mathcal{C}$ is nonempty and vice versa and therefore the integer $m = \max_{\alpha\in\mathcal{C}}|\alpha|$ is well defined such that

i. There exists a magic subset $A \subset I$ with $|A| = m$ such that $h_A = \langle\psi|p_A\bar{p}_{\bar{A}}\rangle \neq 0$ for some single qudit states $|\bar{p}_k\rangle_k$ orthogonal to $|p_k\rangle_k$ for each $k \in \bar{A}$ with $|\bar{p}\rangle_{\bar{A}} = \otimes_{k\in\bar{A}}|\bar{p}_k\rangle_k$;

ii. For every subset $B \subset I$ with $m < |B| < n$ it holds $\langle\psi|p_B\phi_{\bar{B}}\rangle = 0$ for all single qudit states $|\phi_k\rangle_k$ orthogonal to $|p_k\rangle_k$ for each $k \in \bar{B}$ with $|p_B\rangle = \otimes_{k\in B}|p_k\rangle_k$ and $|\phi_{\bar{B}}\rangle = \otimes_{k\in\bar{B}}|\phi_k\rangle_k$.

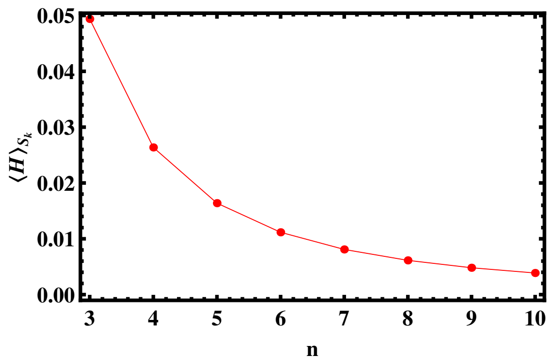

Figure 1: (color online). Violation of Hardy's inequality for W states (k=1) with $3 \leq n \leq 10$.

Also we have $m \leq n-2$ because if there were $k \in I$ such that $h_{\bar{k}} = \langle \psi | p_{\bar{k}} \phi_k \rangle \neq 0$ for some qudit state $|\phi\rangle_k$ orthogonal to $|p_k\rangle_k$, then we would have $|\langle \psi | p_{\bar{k}} \phi'_k \rangle|^2 = |h_I|^2 + |h_{\bar{k}}|^2 > |h_I|^2$ with normalized state $|\phi'\rangle_k \propto h_I^* |p_k\rangle_k + h_{\bar{k}}^* |\phi\rangle_k$, which contradicts the definition of the closest product state as $|p_{\bar{k}} \phi'_k\rangle$ is a product state.

For each qudit $k \in I$ we take $|p_k\rangle_k$ to be $|0\rangle_k$ and for each qubit $k \in \bar{A}$ we regard $|\bar{p}_k\rangle_k$, as it appears in the definition of the magic subset A (item i), to be $|1\rangle_k$ while for each qubit $k \in A$ we take an arbitrary qudit state orthogonal to $|p_k\rangle_k$ to be $|1\rangle_k$. Thus we have picked out two orthogonal states for each qudit with the help of which we can *locally* project n qudits to an n-qubit subspace. Within this local n-qubit subspace we have effectively a set of n-qubits in a projected state (not normalized in general) in its magic basis with a magic subset A satisfying $h_B = 0$ as long as $|A| < |B| < n$. By choosing exactly the same measurement settings as specified in Table 1, we can obtain the desired violation to Hardy's inequality for an arbitrary entangled pure n-qudit state.

5. Examples

5.1 The n-qubit Dicke state

The n-qubit Dicke state

$$|S_k\rangle = \frac{1}{\sqrt{\binom{n}{k}}} \sum_{|\alpha|=k} |0_\alpha 1_{\bar{\alpha}}\rangle \tag{41}$$

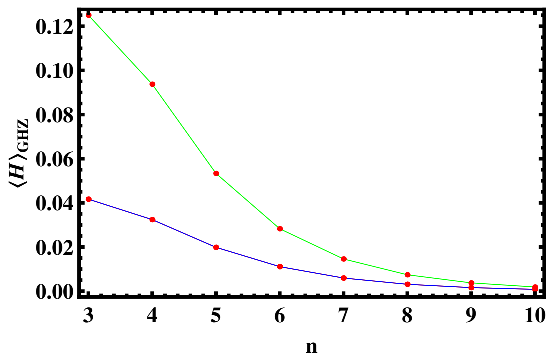

Figure 2: (color online). Violation of Hardy's inequality for generalized GHZ states with $3 \leq n \leq 10$. The green line stands for $\theta = \pi/4$, and the blue line stands for $\theta = \pi/8, 3\pi/8$.

with $0 < k < n$ belongs to the Bell scenario in the magic basis $\{|p_\alpha \bar{p}_{\bar{\alpha}}\rangle\}_{\alpha \subseteq I}$ determined by its closest product state $|p\rangle^{\otimes n}$ with $|p\rangle \propto \sqrt{k}|0\rangle + \sqrt{n-k}|1\rangle$ and $|\bar{p}\rangle$ orthogonal to $|p\rangle$. The magic subset A is any subset of I with $n-2$ elements. Moreover, we have

$$h_I^2 = \binom{n}{k} \frac{k^k (n-k)^{n-k}}{n^n} \tag{42}$$

and $h_A = -h_I/(n-1) < 0$ together with $q = \sqrt{n-1}/n$ and $r = \sqrt{n^2-n+1}/n$, which lead to a violation

$$\langle H \rangle_{S_k} = \frac{h_I^2}{n^2} \tag{43}$$

in the case of degenerate measurement settings $\gamma = 0$. We show the W states ($k = 1$ and $n \geq 3$) as example. Fig. 1 shows the violation of Hardy's inequality for W states with $3 \leq n \leq 10$.

5.2 The generalized GHZ state

The generalized GHZ state

$$|\psi\rangle_{GHZ} \propto h_I^*|0_I\rangle + h_\emptyset^*|1_I\rangle \tag{44}$$

with $h_I h_\emptyset \neq 0$, which is already expanded in a magic basis with the magic subset $A = \emptyset$ and $m = 0$, belongs to the Hardy scenario. By taking $y_0 = 1$ the algebraic equation

$$\langle \bar{b}_I|\psi\rangle \propto |h_\emptyset|^2 + |h_I|^2(-z)^{n-1} = 0 \tag{45}$$

has a nonunital solution

$$z_0 = -e^{i\pi/(n-1)}(|h_\emptyset|/|h_I|)^{2/(n-1)}, \tag{46}$$

which leads to a violation as given in Eq. (39). Let us take $|\psi\rangle_{GHZ} = \cos\theta|0_I\rangle + \sin\theta|1_I\rangle$ as example. We have show the violations of this generalized GHZ state with $\theta = \pi/8, \pi/4$ and $3\pi/8$ in Fig. 2.

5.3 The pure n-qubit state

As the third example the pure n-qubit state

$$|\psi\rangle \propto |0_I\rangle + |0_\alpha 1_{\bar{\alpha}}\rangle + |1_I\rangle \tag{47}$$

with $\alpha = \{1, 2\}$ and $n = 4j + 1$ for $j \geq 1$ belongs to both the Bell and Hardy scenarios. First, the state is expressed already in a magic basis and the magic subset is $A = \alpha$ with $m = 2 < n-2$ since $n \geq 5$. By taking $v = \{n\}$ and $y_0 = 1$ we have $c_k = 0$ for all $k \in A$ and $f = 1, e = -z$. Since $z_0 = i$ is a root of

$$\langle \bar{b}_I|\psi\rangle \propto (1-z)^2(1 + (-z)^{n-3}) = 0 \tag{48}$$

we obtain a violation

$$\langle H\rangle_\psi = \frac{1}{3 \times 2^{n-3}}. \tag{49}$$

Second, if $|0\rangle$ and $|1\rangle$ are exchanged for each qubit then we obtain another magic basis with a magic subset $A = \bar{\alpha}$ with $m = n-2$; i.e., the state $|\psi\rangle$ also belongs to the Bell scenario with a violation

$$\langle H\rangle_\psi = \frac{1}{12}, \tag{50}$$

since $h_{\bar{\alpha}} = h_I = 1/\sqrt{3}$.

5.4 Two-qubit pure states

Consider all the two-qubit pure states, we denote their closest product state as $|00\rangle$. Under this magic basis, all the two-qubit pure states belong to the following single case, the Bell scenario.

In this case, the state can be written as $|\psi\rangle = h_I^*|00\rangle + h_\emptyset^*|11\rangle$ with $h_\emptyset \neq 0$. Bell scenario has a simple violation of Hardy's inequality in Eq. (18) with $\gamma = 0$. If $h_\emptyset = 0$, the state is exactly $|00\rangle$ which is separable and cannot violate any Bell inequalities.

5.5 Three-qubit pure states

Consider all the three-qubit pure states, we denote their closest product state as $|000\rangle$. Under this magic basis, all the three-qubit pure states belong to the following two cases:

i) Bell scenario, i.e. $m = 1$, there exists $A \subset I$ such that $h_A \neq 0$ with $|A| = 1$ and $I = A \cup \bar{A}$. In this case, the state can be written as $|\psi\rangle = h_I^*|000\rangle + h_A^*|011\rangle + \cdots$, where $|011\rangle$ can also be replaced by $|101\rangle$ or $|110\rangle$. Bell scenario has a simple violation of Hardy's inequality in Eq. (18) with $\gamma = 0$.

ii) Hardy scenario, i.e. $m = 0$. In this case, the state can be written as $|\psi\rangle = h_I^*|000\rangle + h_\emptyset^*|111\rangle$ with $h_\emptyset \neq 0$, this state is the three-qubit generalized GHZ state. The violation of Hardy's inequality for the generalized GHZ state has already been shown in the second example. If $h_\emptyset = 0$, the state is exactly $|000\rangle$ which is separable and cannot violate any Bell inequalities.

6. Discussions

In general Bell's inequality is characterized by 3 parameters (n, m, d): each of n observers measures m observables with d outcomes. CHSH inequality belongs to the scenario $(2, 2, 2)$ and it is complete for this scenario. MABK inequality is a generalization of CHSH inequality to multipartite systems but it is not complete for $(n, 2, 2)$ scenario. As a stronger inequality, WWZB inequality is a complete set of Bell's inequalities for n-qubit correlations for $(n, 2, 2)$ scenario. However, even for WWZB inequality, there still exist some entangled pure states which do not violate it. Hardy's inequality also belongs to $(n, 2, 2)$ scenario however it not only contains n-qubit correlations, but also other correlations less than n $(n > 2)$. We have proved Gisin's theorem by using Hardy's inequality and in this sense Hardy's inequality is a more natural generalization of CHSH inequality to multiparticles. This is not surprising because Hardy's argument for nonlocality without inequality is de facto a state-dependent proof of quantum contextuality, which should manifest itself in any quantum state.

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